

Approximation in Müntz spaces $M_{\Lambda,p}$ of L_p functions for $1 < p < \infty$ and bases.

Sergey V. Ludkowski

17 August 2016

Abstract

Müntz spaces satisfying the Müntz and gap conditions are considered. A Fourier approximation of functions in the Müntz spaces $M_{\Lambda,p}$ of L_p functions is studied, where $1 < p < \infty$. It is proved that up to an isomorphism and a change of variables these spaces are contained in Weil-Nagy's class. Moreover, existence of Schauder bases in the Müntz spaces $M_{\Lambda,p}$ is investigated. ¹

1 Introduction.

The immense branch of functional analysis is devoted to topological and geometric properties of topological vector spaces (see, for example, [12, 15, 16, 23]). Studies of bases in Banach spaces compose a large part of it (see, for

¹key words and phrases: Banach space; Müntz space; isomorphism; Schauder basis; Fourier series.

Mathematics Subject Classification 2010: 46B03; 46B15; 46B20; 42A10; 42A20

Acknowledgement: the author was partially supported by DFG project number LU219/10-1

Addresses: Dep. of Mathematics, Paderborn University,
Warburger str. 100, Paderborn D-33095, Germany
and Department of Applied Mathematics,
Moscow State Technical University MIREA,
av. Vernadsky 78, Moscow 119454, Russia
sludkowski@mail.ru

example, [12, 14, 18]-[22, 32] and references therein). It is not surprising that for concrete classes of Banach spaces many open problems remain, particularly for the Müntz spaces $M_{\Lambda,p}$, where $1 < p < \infty$ (see [1]-[6], [10, 30] and references therein). These spaces are defined as completions of the linear span over \mathbf{R} or \mathbf{C} of monomials t^λ with $\lambda \in \Lambda$ on the segment $[0, 1]$ relative to the L_p norm, where $\Lambda \subset [0, \infty)$, $t \in [0, 1]$. In his classical work K. Weierstrass had proved in 1885 the theorem about polynomial approximations of continuous functions on the segment. But the space of continuous functions also forms the algebra. Generalizations of such spaces were considered by C. Müntz in 1914 such that his spaces had not the algebra structure. The problem was whether they have bases. Then a progress was for lacunary Müntz spaces satisfying the condition $\lim_{n \rightarrow \infty} \lambda_{n+1}/\lambda_n > 1$ with a countable set Λ , but in its generality this problem was not solved [10]. It is worth to mention that the system $\{t^\lambda : \lambda \in \Lambda\}$ itself does not contain a Schauder basis for a nonlacunary set Λ satisfying the Müntz and gap conditions.

In section 2 Müntz spaces satisfying the Müntz and gap conditions are considered. A Fourier approximation of functions in the Müntz spaces $M_{\Lambda,p}$ of L_p functions is studied, where $1 < p < \infty$. Necessary definitions are recalled. It is proved that up to an isomorphism and a change of variables these spaces are contained in Weil-Nagy's class. For this purpose in Lemmas 3 and 4, Theorem 5 and Corollary 9 some isomorphisms of Müntz spaces are given. Then in Theorem 13 a relation between Müntz spaces and Weil-Nagy's classes is established. Moreover, existence of Schauder bases in the Müntz spaces $M_{\Lambda,p}$ is investigated in Theorem 16 with the help of Fourier series approximation (see Lemma 14). There is proved that under the Müntz condition and the gap condition Schauder bases exist in the Müntz spaces $M_{\Lambda,p}$, where $1 < p < \infty$.

All main results of this paper are obtained for the first time. They can be used for further investigations of function approximations and geometry of Banach spaces. It is important not only for development of mathematical analysis and functional analysis, but also in their many-sided applications.

2 Approximation in Müntz L_p spaces.

To avoid misunderstandings we first remind necessary definitions and notations.

1. Notation. Let $C([\alpha, \beta], \mathbf{F})$ denote the Banach space of all continuous functions $f : [\alpha, \beta] \rightarrow \mathbf{F}$ supplied with the absolute maximum norm

$$\|f\|_C := \max\{|f(x)| : x \in [\alpha, \beta]\},$$

where $-\infty < \alpha < \beta < \infty$, \mathbf{F} is either the real field \mathbf{R} or the complex field \mathbf{C} .

Then $L_p([\alpha, \beta], \mathbf{F})$ denotes the Banach space of all Lebesgue measurable functions $f : [\alpha, \beta] \rightarrow \mathbf{F}$ possessing the finite norm

$$\|f\|_{L_p([\alpha, \beta], \mathbf{F})} := \left(\int_{\alpha}^{\beta} |f(x)|^p dx \right)^{1/p} < \infty,$$

where $1 \leq p < \infty$ is a marked number, $\alpha < \beta$.

Suppose that $Q = (q_{n,k})$ is a lower triangular infinite matrix with real matrix elements $q_{n,k}$ satisfying the restrictions: $q_{n,k} = 0$ for each $k > n$, where k, n are nonnegative integers. To each 1-periodic function $f : \mathbf{R} \rightarrow \mathbf{R}$ in the space $L_p((\alpha, \alpha + 1), \mathbf{F})$ or in $C_0([\alpha, \alpha + 1], \mathbf{F}) := \{f : f \in C([\alpha, \alpha + 1], \mathbf{F}), f(\alpha) = f(\alpha + 1)\}$ is posed a trigonometric polynomial

$$(1) \quad U_n(f, x, Q) := \frac{a_0}{2} q_{n,0} + \sum_{k=1}^n q_{n,k} (a_k \cos(2\pi kx) + b_k \sin(2\pi kx)),$$

where $a_k = a_k(f)$ and $b_k = b_k(f)$ are the Fourier coefficients of a function $f(x)$.

For measurable 1-periodic functions h and g their convolution is defined whenever it exists by the formula:

$$(2) \quad (h * g)(x) := 2 \int_{\alpha}^{\alpha+1} h(x-t)g(t)dt.$$

Putting the kernel of the operator U_n to be:

$$(3) \quad U_n(x, Q) := \frac{q_{n,0}}{2} + \sum_{k=1}^n q_{n,k} \cos(2\pi kx)$$

we get

$$(4) \quad U_n(f, x, Q) = (f * U_n(\cdot, Q))(x) = (U_n(\cdot, Q) * f)(x).$$

The norms of these operators are:

$$(5) \quad L_n(Q, E) := \sup_{f \in E, \|f\|_E=1} \|U_n(f, x, Q)\|_E,$$

which are constants of a summation method, where $\|*\|_E$ denotes a norm on a Banach space E , where either $E = C_0([\alpha, \alpha+1], \mathbf{F})$ or $E = L_p((\alpha, \alpha+1), \mathbf{F})$ with $1 \leq p < \infty$, while $\alpha \in \mathbf{R}$ is a marked real number.

As usually $\text{span}_{\mathbf{F}}(v_k : k)$ will stand for the linear span of vectors v_k over a field \mathbf{F} .

Henceforward the Fourier summation methods prescribed by sequences of operators $\{U_m : m\}$ which converge on E

$$(6) \quad \lim_{m \rightarrow \infty} U_m(f, x, Q) = f(x)$$

in the E norm will be considered.

2. Definition. Take a countable infinite subset $\Lambda = \{\lambda_k : k \in \mathbf{N}\}$ in the set $(0, \infty)$ so that $\{\lambda_k : k \in \mathbf{N}\}$ is a strictly increasing sequence.

Henceforth it is supposed that the set Λ satisfies the gap condition

$$(1) \quad \inf_k \{\lambda_{k+1} - \lambda_k\} =: \alpha_0 > 0 \text{ and the Müntz condition}$$

$$(2) \quad \sum_{k=1}^{\infty} \frac{1}{\lambda_k} =: \alpha_1 < \infty.$$

The completion of the linear space containing all monomials at^λ with $a \in \mathbf{F}$ and $\lambda \in \Lambda$ and $t \in [\alpha, \beta]$ relative to the L_p norm is denoted by $M_{\Lambda,p}([\alpha, \beta], \mathbf{F})$, where $0 \leq \alpha < \beta < \infty$, $1 \leq p$, also by $M_{\Lambda,C}([\alpha, \beta], \mathbf{F})$ when it is completed relative to the $\|\cdot\|_C$ norm. Shortly they will also be written as $M_{\Lambda,p}$ or $M_{\Lambda,C}$ respectively for $\alpha = 0$ and $\beta = 1$, when \mathbf{F} is specified.

Before subsections about the Fourier approximation in Müntz spaces auxiliary Lemmas 3, 4 and Theorem 5 are proved about isomorphisms of Müntz spaces M_{Λ,L_p} . With the help of them our consideration reduces to a subclass of Müntz spaces M_{Λ,L_p} so that a set Λ is contained in the set of natural numbers \mathbf{N} .

3. Lemma. *For each $0 < \delta < 1$ the Müntz spaces $M_{\Lambda,p}([0, 1], \mathbf{F})$ and $M_{\Lambda,p}([\delta, 1], \mathbf{F})$ are linearly topologically isomorphic, where $1 \leq p < \infty$.*

Proof. For every $0 < \delta < 1$ and $0 < \epsilon \leq 1$ and $f \in E := L_p([0, 1], \mathbf{F})$ the norms $\|f\|_{E[0,1]}$ and $\epsilon\|f|_{[0,\delta]}\|_{E[0,\delta]} + \|f|_{[\delta,1]}\|_{E[\delta,1]}$ are equivalent, where

$E[\alpha, \beta] := E \cap L_p([\alpha, \beta], \mathbf{F})$ for $0 \leq \alpha < \beta \leq 1$. Due to the Remez-type and the Nikolski-type inequalities (see Theorem 6.2.2 in [3] and Theorem 7.4 in [4]) for each Λ satisfying the Müntz condition there is a constant $\eta > 0$ so that $\|h|_{[0, \delta]}\|_{E[0, \delta]} \leq \eta \|h|_{[\delta, 1]}\|_{E[\delta, 1]}$ for each $h \in M_{\Lambda, p}$, where η is independent of h . Therefore the norms $\|h|_{[\delta, 1]}\|_{E[\delta, 1]}$ and $\|h\|_{E[0, 1]}$ are equivalent on $M_{\Lambda, p}[0, 1]$. Certainly each polynomial $a_1 t^{\lambda_1} + \dots + a_n t^{\lambda_n}$ defined on the segment $[\delta, 1]$ has the natural extension on $[0, 1]$, where $a_1, \dots, a_n \in \mathbf{F}$ are constants and t is a variable. Thus the Müntz spaces $M_{\Lambda, p}[0, 1]$ and $M_{\Lambda, p}[\delta, 1]$ are linearly topologically isomorphic as normed spaces for each $0 < \delta < 1$.

4. Lemma. *The Müntz spaces $M_{\Lambda, p}$ and $M_{\Xi \cup (\alpha\Lambda + \beta), p}$ are linearly topologically isomorphic for every $\beta \geq 0$ and $\alpha > 0$ and a finite subset Ξ in $(0, \infty)$, where $1 \leq p < \infty$.*

Proof. We have that a sequence $\{\lambda_k : k \in \mathbf{N}\}$ is strictly increasing and satisfies the gap condition and hence $\lim_n \lambda_n = \infty$. We order a set $\Xi \cup (\alpha\Lambda + \beta)$ into a strictly increasing sequence also.

In virtue of Theorem 9.1.6 [10] the Müntz space $M_{\Lambda, p}$ contains a complemented isomorphic copy of l_p , consequently, $M_{\Lambda, p}$ and $M_{\Xi \cup \Lambda, p}$ are linearly topologically isomorphic as normed spaces.

Then from Lemma 3 taking $\alpha > 0$ we deduce that

$$(1) \quad \int_{\delta}^1 |f(t)|^p dt = \alpha \int_{\delta^{1/\alpha}}^1 |f(x^\alpha)|^p x^{(\alpha-1)} dx \leq \alpha \max(1, \delta^{(1-\alpha^{-1})}) \int_{\delta^{1/\alpha}}^1 |f(x^\alpha)|^p dx$$

and

$$(2) \quad \int_{\delta}^1 |f(x^\alpha)|^p dx = \alpha^{-1} \int_{\delta^\alpha}^1 |f(t)|^p t^{(\alpha^{-1}-1)} dt \leq \alpha^{-1} \max(1, \delta^{(1-\alpha)}) \int_{\delta^\alpha}^1 |f(t)|^p dt$$

for each $f \in M_{\Lambda, p}$ and hence $M_{\alpha\Lambda, p}$ is isomorphic with $M_{\Lambda, p}$. Considering the set $\Lambda_1 = \Lambda \cup \{\frac{\beta}{\alpha}\}$ and then the set $\alpha\Lambda_1$ we get that $M_{\Lambda, p}$ and $M_{\alpha\Lambda + \beta, p}$ are linearly topologically isomorphic as normed spaces as well.

5. Theorem. *Let increasing sequences $\Lambda = \{\lambda_n : n\}$ and $\Upsilon = \{v_n : n\}$ of positive numbers satisfy Conditions 2(1, 2) and let $\lambda_n \leq v_n$ for each n . If $\sup_n (v_n - \lambda_n) = \delta$, where $\delta < (8 \sum_{n=1}^{\infty} \lambda_n^{-1})^{-1}$, then $M_{\Lambda, p}$ and $M_{\Upsilon, p}$ are the isomorphic Banach spaces, where $1 \leq p < \infty$.*

Proof. There exist the natural isometric linear embeddings of the Müntz spaces $M_{\Lambda, p}$ and $M_{\Upsilon, p}$ into $M_{\Lambda \cup \Upsilon, p}$. We choose a sequence of sets Υ_k satisfying the following restrictions

(1) $\Upsilon_k = \{v_{k,n} : n = 1, 2, \dots\} \subset \Lambda \cup \Upsilon$ and $v_{k,n} \in \{\lambda_n, v_n\}$ for each $k = 0, 1, 2, \dots$ and $n = 1, 2, \dots$, where $\Upsilon_0 = \Lambda$;

(2) $v_{k,n} \leq v_{k+1,n}$ for each $k = 0, 1, 2, \dots$ and $n = 1, 2, \dots$;

(3) $\{\Delta_{k+1,n} : n = 1, 2, \dots\}$ is a monotone decreasing subsequence tending to zero (may be finite or infinite) with positive terms $\Delta_{k+1,n}$ obtained from the sequence $\delta_{k+1,j} := v_{k+1,j} - v_{k,j}$ by elimination of zero terms. Denote by $\theta = \theta_{k+1} : \{j : j \in \mathbf{N}, \delta_{k+1,j} \neq 0\} \rightarrow \mathbf{N}$ the corresponding enumeration mapping such that $\Delta_{k+1,\theta(j)} = \delta_{k+1,j}$ for each $j \in \mathbf{N}$ so that $\delta_{k+1,j} \neq 0$ is not zero;

(4) $\{m(k+1) : k\}$ is a monotone increasing sequence with $m(k+1) := \min\{n : v_n - v_{k+1,n} \neq 0; \forall l < n \ v_l = v_{k+1,l}\}$.

Let $f \in M_{\Upsilon_k,p}$. In view of Theorem 6.2.3 and Corollary 6.2.4 [10] a function f has a power series expansion

$$f(z) = \sum_{n=1}^{\infty} a_n z^{v_{k,n}} \text{ on } [0, 1),$$

where $a_n \in \mathbf{F}$ for each $n \in \mathbf{N}$.

Therefore, for each $f \in M_{\Upsilon_k,p}$ we consider the power series $f_1(t) = \sum_{n=1}^{\infty} a_n t^{v_{k+1,n}}$, where the power series decomposition $f(t) = \sum_{n=1}^{\infty} a_n t^{v_{k,n}}$ converges for each $0 \leq t < 1$, since f is analytic on $[0, 1)$. Then we infer that

$$f(t^2) - f_1(t^2) = \sum_{n=1}^{\infty} a_n t^{v_{k,n}} u_{\theta(n)}(t) \text{ with } u_{\theta(n)}(t) := t^{v_{k,n}} - t^{v_{k,n} + 2\Delta_{k+1,\theta(n)}}$$

so that $u_l(t)$ is a monotone decreasing sequence by l and hence

$$|f(t^2) - f_1(t^2)| \leq 2|u_{\theta(m(k+1))}(t)| |f(t)|$$

according to Dirichlet's criterium (see, for example, [8]) for each $0 \leq t < 1$, where $\theta = \theta_{k+1}$. Therefore, the function $f_1(t)$ is analytic on $[0, 1)$ and

$$(5) \quad \|f - f_1\|_{L_p([0,1],\mathbf{F})} \leq 2^{2+1/p} \|f\|_{L_p([0,1],\mathbf{F})} \Delta_{k+1,\theta(m(k+1))} / \lambda_{m(k+1)},$$

since the mapping $t \mapsto t^2$ is the orientation preserving diffeomorphism of $[0, 1]$ onto itself, also $|u_{m(k+1)}(t)| \leq 2\Delta_{k+1,\theta(m(k+1))} / \lambda_{m(k+1)}$ for each $0 \leq t \leq 1$ by Lemma 7.3.1 [10] and

$$\begin{aligned} \|f - f_1\|_{L_p([0,1],\mathbf{F})} &= \left[\int_0^1 |f(\tau) - f_1(\tau)|^p d\tau \right]^{1/p} \\ &= \left[2 \int_0^1 |f(t^2) - f_1(t^2)|^p t dt \right]^{1/p} \leq \left[2^{p+1} \int_0^1 |u_{m(k+1)}(t)|^p |f(t)|^p t dt \right]^{1/p} \end{aligned}$$

$$\leq 2^{2+1/p} [\int_0^1 |f(t)|^p dt]^{1/p} \Delta_{k+1, \theta(m(k+1))} / \lambda_{m(k+1)}.$$

Thus the series $\sum_{n=1}^{\infty} a_n t^{v_{k+1,n}}$ converges on $[0, 1)$.

Inequality (5) implies that the linear isomorphism T_k of $M_{\Upsilon_k, p}$ with $M_{\Upsilon_{k+1}, p}$ exists such that $\|T_k - I\| \leq 2^{2+1/p} \Delta_{k+1, \theta(m(k+1))} / \lambda_{m(k+1)}$, $T_k : M_{\Upsilon_k, p} \rightarrow M_{\Upsilon_{k+1}, p}$. Then we take the sequence of operators $S_n := T_n T_{n-1} \dots T_0 : M_{\Lambda, p} \rightarrow M_{\Upsilon_{n+1}, p} \subset M_{\Lambda \cup \Upsilon, p}$. The space $M_{\Lambda \cup \Upsilon, p}$ is complete and the sequence $\{S_n : n\}$ operator norm converges to an operator $S : M_{\Lambda, p} \rightarrow M_{\Lambda \cup \Upsilon, p}$ so that $\|S - I\| < 1$, since

$$\sum_{k=0}^{\infty} \Delta_{k+1, \theta(m(k+1))} / \lambda_{m(k+1)} \leq \delta \sum_{n=1}^{\infty} \lambda_n^{-1} < 1/8$$

and $p \geq 1$, where I denotes the unit operator. Therefore, the operator S is invertible. On the other hand, from Conditions (1 – 4) it follows that $S(M_{\Lambda, p}) = M_{\Upsilon, p}$.

Now we recall necessary definitions and notations of the Fourier approximation theory and then present useful lemmas.

6. Notation. Henceforth \mathbf{F} denotes the set of all pairs (ψ, β) satisfying the conditions: $(\psi(k) : k \in \mathbf{N})$ is a sequence of non-zero numbers for which $\lim_{k \rightarrow \infty} \psi(k) = 0$ the limit is zero, β is a real number, also

$$(1) \quad \mathcal{D}_{\psi, \beta}(x) := \sum_{k=1}^{\infty} \psi(k) \cos(2\pi kx + \beta\pi/2)$$

is the Fourier series of some function from $L_1[0, 1]$. By \mathbf{F}_1 is denoted the family of all positive sequences $(\psi(k) : k \in \mathbf{N})$ tending to zero with $\Delta_2 \psi(k) := \psi(k-1) - 2\psi(k) + \psi(k+1) \geq 0$ for each k so that the series

$$(2) \quad \sum_{k=1}^{\infty} \frac{\psi(k)}{k} < \infty$$

converges. The set of all downward convex functions $\psi(v)$ for each $v \geq 1$ so that $\lim_{v \rightarrow \infty} \psi(v) = 0$ is denoted by \mathcal{M} , while \mathcal{M}_1 is its subset of functions satisfying Condition (2).

Then

$$(3) \quad \rho_n(f, x) := f(x) - S_{n-1}(f, x)$$

is the approximation precision of f by the Fourier series $S(f, x)$, where

$$(4) \quad S_n(f, x) := \frac{a_0}{2} + \sum_{k=1}^n (a_k \cos(2\pi kx) + b_k \sin(2\pi kx))$$

is the partial Fourier sum approximating a Lebesgue integrable 1-periodic function f on $(0, 1)$.

7. Definition. Suppose that $f \in L_1(\alpha, \alpha + 1)$ and $S[f]$ is its Fourier series with coefficients $a_k = a_k(f)$ and $b_k = b_k(f)$, while $\psi(k)$ is an arbitrary sequence real or complex. If the function

$$D_\beta^\psi f := f_\beta^\psi := \sum_{k=1}^{\infty} [a_k(f) \cos(2\pi kx + \beta\pi/2) + b_k(f) \sin(2\pi kx + \beta\pi/2)] / \psi(k)$$

belongs to the space $L(\alpha, \alpha + 1)$ of all Lebesgue integrable (summable) functions on $(\alpha, \alpha + 1)$, then f_β^ψ is called the Weil (ψ, β) derivative of f . Then $L_\beta^\psi = L_\beta^\psi(\alpha, \alpha + 1)$ stands for the family of all functions $f \in L(\alpha, \alpha + 1)$ with $f_\beta^\psi \in L(\alpha, \alpha + 1)$, we also put $L_{\beta,p}^\psi := \{f : f \in L_\beta^\psi, \|f_\beta^\psi\|_{L_p(\alpha, \alpha+1)} \leq 1\}$. Particularly, for $\psi(k) = k^{-r}$ this space L_β^ψ is Weil-Nagy's class $W_\beta^r = W_\beta^r(\alpha, \alpha + 1)$ and the notation $W_{\beta,p}^r$ can be used instead of $L_{\beta,p}^\psi$ in this case. Put particularly $W_\beta^r L_p(\alpha, \alpha + 1) := \{f : f \in L_p(\alpha, \alpha + 1), \exists f_\beta^\psi \in L_p(\alpha, \alpha + 1)\}$, where $1 < p < \infty$.

Then let $\mathcal{E}_n(X) := \sup\{\|\rho_n(f; x)\|_{L_p(\alpha, \alpha+1)} : f \in X\}$,

$E_n(f)_p := \inf\{\|f - T_{n-1}\|_{L_p(\alpha, \alpha+1)} : T_{n-1} \in \mathcal{T}_{2n-1}\}$,

$E_n(X) := \sup\{E_n(f)_p : f \in X\}$,

where X is a subset in $L_p(\alpha, \alpha + 1) = L_p((\alpha, \alpha + 1), \mathbf{R})$,

$$\mathcal{T}_{2n-1} := \{T_{n-1}(x) = \frac{c_0}{2} + \sum_{k=1}^{n-1} (c_k \cos(2\pi kx) + d_k \sin(2\pi kx)); c_k, d_k \in \mathbf{R}\}$$

denotes the family of all trigonometric polynomials T_{n-1} of degree not greater than $n - 1$.

8. Lemma. Suppose that $Q_\alpha f(t) := f(t^\alpha)$ for each $f : [0, 1] \rightarrow \mathbf{F}$, where $0 < \alpha$, $t \in [0, 1]$, $1 < p < \infty$. Then for each $1 < \alpha < \infty$ there exists $0 < \delta < 1$ such that the operator Q_α from $L_p(\delta^\alpha, 1)$ into $L_p(\delta, 1)$ has the norm $\|Q_\alpha\| < 1$.

Proof. The Banach spaces $L_p(\delta, 1)$ and $L_p(\delta^\alpha, 1)$ are defined with the help of the Lebesgue measure on \mathbf{R} . Then Formula 4(2) implies that $\|Q_\alpha\| < 1$ as soon as $\alpha^{-1} \max(1, \delta^{(1-\alpha)}) < 1$. That is when $\{\delta^{(1-\alpha)} < \alpha\} \iff \{\ln \delta > (1 - \alpha)^{-1} \ln \alpha\}$, since $\alpha > 1$ and $0 < \delta < 1$.

9. Corollary. Let $1 < \alpha < \infty$ and $0 < \delta < 1$ so that $\delta > \alpha^{1/(1-\alpha)}$, let also $Z_{\Lambda,p,\alpha,\delta} := (I - Q_\alpha)[M_{\Lambda,p}(\delta^\alpha, 1)]$, where $1 < p < \infty$, while I is the unit

operator. Then $Z_{\Lambda,p,\alpha,\delta}$ is isomorphic with $M_{\Lambda,p}(\delta^\alpha, 1)$.

Proof. There is the natural embedding of $L_p(a, b)$ into $L_p(c, d)$ when $c \leq a$ and $b \leq d$ such that $f \mapsto f\chi_{(a,b)}$ for each $f \in L_p(a, b)$, where χ_A notates the characteristic function of a set A . Since $\|Q_\alpha\| < 1$, then the operator $I - Q_\alpha$ is invertible (see [13]).

10. Lemma. Let $f \in L_p(0, 1)$, where $1 < p < \infty$. Then

$$\lim_{\eta \downarrow 0} \eta^{-1/q} \int_{1-\eta}^1 f(t) dt = 0,$$

where $1/q + 1/p = 1$.

Proof. Since $f \in L_p(0, 1)$, then $|f(t)|^p \mu(dt)$ is a σ -additive and finite measure on $(0, 1)$, where μ is the Lebesgue measure on \mathbf{R} (see, for example, [7], Theorems V.5.4.3 and V.5.4.5 [13]). Therefore, the limit exists

$$(1) \quad \lim_{\eta \downarrow 0} \int_{1-\eta}^1 |f(t)|^p dt = 0.$$

From Holder's inequality it follows that

$$\begin{aligned} \left| \int_{1-\eta}^1 f(t) dt \right| &\leq \left(\int_{1-\eta}^1 |f(t)|^p dt \right)^{1/p} \left(\int_{1-\eta}^1 1 dt \right)^{1/q} \\ &= \eta^{1/q} \left(\int_{1-\eta}^1 |f(t)|^p dt \right)^{1/p} \text{ hence} \\ (2) \quad \left| \eta^{-1/q} \int_{1-\eta}^1 f(t) dt \right| &\leq \left(\int_{1-\eta}^1 |f(t)|^p dt \right)^{1/p}. \end{aligned}$$

Thus from Formulas (1) and (2) the statement of this lemma follows.

11. Note. We remind the following definition: the family of all Lebesgue measurable functions $f : (a, b) \rightarrow \mathbf{R}$ satisfying the condition

$$\|f\|_{L_{s,w}(a,b)} := \sup_{y>0} (y^s \mu\{t : t \in (a, b), |f(t)| \geq y\})^{1/s} < \infty$$

is called the weak L_s space and denoted by $L_{s,w}(a, b)$, where μ notates the Lebesgue measure on the real field \mathbf{R} , $0 < s < \infty$, $(a, b) \subset \mathbf{R}$ (see, for example, §9.5 [7], §IX.4 [26], [29]).

The following proposition 12 is used below in theorem 13 to prove that functions of Müntz spaces $M_{\Lambda,p}$ for Λ satisfying the Müntz condition and the gap condition belong to Weil-Nagy's class, where $1 < p < \infty$.

12. Proposition. Suppose that an increasing sequence $\Lambda = \{\lambda_n : n\}$ of natural numbers satisfies the Müntz condition, $1 < p < \infty$ and $f \in M_{\Lambda,p}$.

Then $dh(x)/dx \in L_{s,w}(0,1)$ for a function $h(x) = f(x) - f(x^2)$, where $s = p/(p+1)$.

Proof. In view of Theorem 6.2.3 and Corollary 6.2.4 [10] a function f is analytic on $(0,1)$ and consequently, h is analytic on $(0,1)$, hence a derivative $dh(x)/dx$ is also analytic on $(0,1)$. Moreover, the series

$$(1) f(z) = \sum_{n=1}^{\infty} a_n z^{\lambda_n}$$

converges on $\dot{B}_1(0)$, where $\dot{B}_r(x) := \{y : y \in \mathbf{C}, |y-x| < r\}$ denotes the open disk in \mathbf{C} of radius $r > 0$ with center at $x \in \mathbf{C}$, where $a_n \in \mathbf{F}$ is an expansion coefficient for each $n \in \mathbf{N}$. That is, the functions f and h have holomorphic univalent extensions on $\dot{B}_1(0)$, since $\Lambda \subset \mathbf{N}$ (see Theorem 20.5 in [27]). Take the function $H(x) = \int_x^1 h(t)dt$, where $x \in [0,1]$. In virtue of Theorem VI.4.2 [13] and Lyapunov's inequality (formula (27) in §II.6 [28]) this function is continuous so that $H(1) = 0$. Together with formula (1) this implies that the function $H(x)$ belongs to $M_{\{0\} \cup (\Lambda+1), C}$ and has a holomorphic univalent extension on $\dot{B}_1(0)$.

Then we put $g(z) = (1-z)^{-1/q}H(z)$ for each $z \in \dot{B}_1(0)$, where $1/q+1/p = 1$. From Lemma 10 it follows that

$$(2) \lim_{z \rightarrow 1} g(z) = 0.$$

Thus the function $g(z)$ is holomorphic (may be multivalent because of the multiplier $(1-z)^{-1/q}$) on $\dot{B}_1(0)$ and continuous on $\dot{B}_1(0) \cup \{1\}$.

According to Cauchy's formula 21(5) in [27]

$$(3) \quad h'(z) = -\frac{1}{\pi i} \int_{\gamma} \frac{H(y)}{(y-z)^3} dy$$

for each $z \in \dot{B}_{1/2}(1/2)$, where γ is an oriented rectifiable boundary $\gamma = \partial G$ of a simply connected open domain G contained in $\dot{B}_{1/2}(1/2)$ such that $z \in G$. Particularly, this is valid for each z in $(1/2, 1)$ and $G = \dot{B}_{1/2}(1/2)$.

On the other hand, the function $g(z)$ is bounded on $B_{1/2}(1/2)$, where $B_r(x) := \{y : y \in \mathbf{C}, |y-x| \leq r\}$ notates the closed disk of radius $r > 0$ with center at $x \in \mathbf{C}$. Thus $K = \sup_{z \in B_{1/2}(1/2)} |g(z)| < \infty$. Estimating the integral (3) and taking into account formula (2) we infer that $|h'(t)| \leq 2K/(1-t)^{1+1/p}$ for each $t \in (3/4, 1)$, since $1/q + 1/p = 1$. Together with the analyticity of h' on $[0, 1)$ this implies that

$$\sup_{y>0} (y^s \mu\{t : t \in (a, b), |h'(t)| \geq y\})^{1/s} < \infty,$$

where $s = p/(p+1)$. Thus $h' \in L_{s,w}(0,1)$.

13. Theorem. *Let an increasing sequence $\Lambda = \{\lambda_n : n\}$ of natural numbers satisfy the Müntz condition, also $1 > \delta > 1/2$ and $1 < p < \infty$ and let $\sigma(x) = \delta^2 + x(1 - \delta^2)$, where $0 \leq x \leq 1$. Then for each $0 < \gamma < 1$ there exists $\beta = \beta(\gamma) \in \mathbf{R}$ so that $Z_{\Lambda,p,2,\delta} \circ \sigma \subset W_\beta^\gamma L_p(0,1)$.*

Proof. Let $f \in M_{\Lambda,p}(0,1)$ and $v(x) = (I - Q_2)f(\sigma(x))$, then $v(x)$ is analytic on $(0,1)$, since f is analytic on $(0,1)$ and $\sigma[0,1] = [\delta^2, 1]$. We take its 1-periodic extension v_0 on \mathbf{R} .

According to Proposition 1.7.2 [31] (or see [34]) $h \in W_\beta^\gamma L_p(0,1)$ if and only if there exists a function $\phi = \phi_{h,\gamma,\beta}$ which is 1-periodic on \mathbf{R} and Lebesgue integrable on $[0,1]$ such that

$$(1) \quad h(x) = \frac{a_0(h)}{2} + (\phi * \mathcal{D}_{\psi,\beta})(x),$$

where $a_0(h) = 2 \int_0^1 h(t)dt$ (see §§6 and 7).

We take a sequence $U_n(t, Q)$ given by Formula 1(3) so that

$$\lim_m q_{m,k} = 1 \text{ for each } k \text{ and } \sup_m L_m(Q, L_p) < \infty \text{ and } \sup_{m,k} |q_{m,k}| < \infty$$

and write for short $U_n(t)$ instead of $U_n(t, Q)$. Under these conditions the limit exists

$$(2) \quad \lim_n (v * U_n)(x) = v(x)$$

in $L_p(0,1)$ norm for each $v \in L_p((0,1), \mathbf{F})$ according to Chapters 2 and 3 in [31] (see also [2, 34]).

On the other hand, Formula I(10.1) [31] provides

$$(3) \quad S[(y_{\beta_1}^{\psi_1})_{\beta_2 - \beta_1}^{\psi_2/\psi_1}] = S[y_{\beta_2}^{\psi_2}],$$

where $S[y]$ is the Fourier series corresponding to a function $y \in L_{\beta_2}^{\psi_2}$, when $(\psi_1, \bar{\beta}_1) \leq (\psi_2, \bar{\beta}_2)$.

Put $\theta(k) = k^{\gamma-1}$ for all $k \in \mathbf{N}$. Then $\mathcal{D}_{\theta,-\beta} \in L_1(0,1)$ for each $\beta \in \mathbf{R}$ due to Theorems II.13.7, V.1.5 and V.2.24 [34] (or see [2]). This is also seen from chapters I and V in [31] and Formulas (1) and (3) above. In view of Dirichlet's theorem (see §430 in [8]) the function $\mathcal{D}_{\theta,-\beta}(x)$ is continuous on the segment $[\delta, 1 - \delta]$ for each $0 < \delta < 1/4$.

According to formula 2.5.3.(10) in [25]

$$\int_0^\infty x^{\alpha-1} \left(\frac{\sin(bx)}{\cos(bx)} \right) dx = b^{-\alpha} \Gamma(\alpha) \left(\frac{\sin(\pi\alpha/2)}{\cos(\pi\alpha/2)} \right)$$

for each $b > 0$ and $0 < \operatorname{Re}(\alpha) < 1$. On the other hand, the integration by parts gives:

$$\int_a^\infty x^{\alpha-1} \begin{pmatrix} \sin(bx) \\ \cos(bx) \end{pmatrix} dx = b^{-1} a^{\alpha-1} \begin{pmatrix} \cos(ab) \\ -\sin(ab) \end{pmatrix} - b^{-1}(\alpha-1) \int_a^\infty x^{\alpha-2} \begin{pmatrix} -\cos(bx) \\ \sin(bx) \end{pmatrix} dx$$

for every $a > 0$, $b > 0$ and $0 < \operatorname{Re}(\alpha) < 1$. From formulas V(2.1), theorems V.2.22 and V.2.24 in [34] (see also [5, 24]) we infer the asymptotic expansions

$$\begin{aligned} \sum_{n=1}^\infty n^{-\alpha} \sin(2\pi nx) &\approx (2\pi x)^{\alpha-1} \Gamma(1-\alpha) \cos(\pi\alpha/2) + \mu x^\alpha, \\ \sum_{n=1}^\infty n^{-\alpha} \cos(2\pi nx) &\approx (2\pi x)^{\alpha-1} \Gamma(1-\alpha) \sin(\pi\alpha/2) + \nu x^\alpha \end{aligned}$$

in a small neighborhood $0 < x < \delta$ of zero, where $0 < \delta < 1/4$, $0 < \alpha < 1$, μ and ν are real constants. Taking $\beta = \alpha = 1 - \gamma$ we get that $\mathcal{D}_{\theta, -\beta}(x) \in L_\infty(0, 1)$.

Evidently, for Lebesgue measurable functions $f : \mathbf{R} \rightarrow \mathbf{R}$ and $g : \mathbf{R} \rightarrow \mathbf{R}$ there is the equality $\int_{-\infty}^\infty f(x-t)\chi_{[0,\infty)}(x-t)g(t)\chi_{[0,\infty)}(t)dt = \int_0^x f(x-t)g(t)dt$ for each $x > 0$ whenever this integral exists, where χ_A denotes the characteristic function of a subset A in \mathbf{R} such that $\chi_A(y) = 1$ for each $y \in A$, also $\chi_A(y) = 0$ for each y outside A , $y \in \mathbf{R} \setminus A$. Particularly, if $0 < x \leq T$, where $0 < T < \infty$ is a constant, then $\int_0^x f(x-t)g(t)dt = \int_0^\infty f(x-t)\chi_{[0,T]}(x-t)g(t)\chi_{[0,T]}(t)dt$ (see also [8, 13]). This is applicable to formula 1(2) putting $\alpha = 0$ there and with the help of the equality

$$\int_0^1 f(x-t)g(t)dt = \int_0^x f(x-t)g(t)dt + \int_0^{1-x} f_1((1-x)-v)g_1(v)dv$$

for each $0 \leq x \leq 1$ and 1-periodic functions f and g and using also that $\|f|_{[a,b]}\| \leq \|f|_{[0,1]}\| = \|f_1|_{[0,1]}\|$ for the considered here types of norms for each $[a, b] \subset [0, 1]$, where $f_1(t) = f(-t)$ and $g_1(t) = g(-t)$ for each $t \in \mathbf{R}$, since

$$\int_x^1 f(x-t)g(t)dt = \int_0^{1-x} f(v-1+x)g(1-v)dv.$$

Mention that according to the weak Young inequality

$$(4) \quad \|\xi * \eta\|_p \leq K_{r,s} \|\xi\|_r \|\eta\|_{s,w}$$

for each $\xi \in L_r$ and $\eta \in L_{s,w}$, where $1 \leq p, r \leq \infty$, $0 < s < \infty$ and $r^{-1} + s^{-1} = 1 + p^{-1}$, $K_{r,s} > 0$ is a constant independent of ξ and η (see theorem 9.5.1 in [7], §IX.4 in [26]).

In virtue of formula (3), the weak Young inequality (4) and Proposition 12 there exists a function s in $L_p(0, 1)$ so that

$$s(x) = \lim_n ((\mathcal{D}_{\theta, -\beta} * U_n) * v'_0)(x),$$

where $\beta = 1 - \gamma$. Therefore $\phi_{v_0, \gamma, \beta} = s$ and $D_\beta^\psi v_0 = s$ according to (1) and (3). Thus $v_0 \in W_\beta^\gamma L_p(0, 1)$.

Below Lemma 14 and Proposition 15 are given. They are used in subsection 16 for proving existence of a Schauder basis. On the other hand, Theorem 13 is utilized that to prove Lemma 14.

14. Lemma. *If an increasing sequence Λ of natural numbers satisfies the Müntz condition, also $0 < \gamma < 1$ and $1 < p < \infty$, $1 > \delta > 1/2$,*

$$X = \{h : h = f \circ \sigma, f \in Z_{\Lambda, p, 2, \delta}; \|f\|_{L_p((\delta^2, 1), \mathbf{R})} \leq 1\},$$

then a positive constant $\omega = \omega(p, \gamma)$ exists so that

$$(1) \quad E_n(X) \leq \mathcal{E}_n(X) \leq \omega n^{-\gamma}$$

for each natural number $n \in \mathbf{N}$.

Proof. Due to Theorem 13 the inclusion is valid $h(x) \in W_\beta^\gamma L_p(0, 1)$ for each $h \in Z_{\Lambda, p, 2, \delta} \circ \sigma$, where ψ is in F_1 so that $\psi(k) = k^{-\gamma}$ for each $k \in \mathbf{N}$, $\beta = 1 - \gamma$. Then $\|h\|_{L_p((0, 1), \mathbf{R})} = (1 - \delta^2)^{-1/p} \|f\|_{L_p((\delta^2, 1), \mathbf{R})} \leq (1 - \delta^2)^{-1/p}$ for each $h \in X$, since

$$(2) \quad \int_0^1 |h(x)|^p dx = (1 - \delta^2)^{-1} \int_{\delta^2}^1 |f(t)|^p dt.$$

Therefore, $X \subset (1 - \delta^2)^{-1/p} W_\beta^\gamma L_p(0, 1)$ (see also §7), where $bY := \{f : f = bg, g \in Y\}$ for a linear space Y over \mathbf{R} and a marked real number b .

Then estimate (1) follows from Theorem V.5.3 in [31].

15. Proposition. *Let X be a Banach space over \mathbf{R} and let Y be its Banach subspace so that they fulfill conditions (1 – 4) below:*

(1) *there is a sequence $(e_i : i \in \mathbf{N})$ in X such that e_1, \dots, e_n are linearly independent vectors and $\|e_n\|_X = 1$ for each n and*

(2) *there exists a Schauder basis $(z_n : n \in \mathbf{N})$ in X such that*

$z_n = \sum_{k=1}^n b_{k,n} e_k$ for each $n \in \mathbf{N}$, where $b_{k,n}$ are real coefficients;

(3) *for every $x \in Y$ and $n \in \mathbf{N}$ there exist $x_1, \dots, x_n \in \mathbf{R}$ so that*

$$\|x - \sum_{i=1}^n x_i e_i\|_X \leq s(n) \|x\|,$$

where $s(n)$ is a strictly monotone decreasing positive function with

$$\lim_{n \rightarrow \infty} s(n) = 0 \text{ and}$$

$$(4) \quad u_n = \sum_{l=m(n)}^{k(n)} u_{n,l} e_l,$$

where $u_{n,l} \in \mathbf{R}$ for each natural numbers k and l , where a sequence $(u_n : n \in$

\mathbf{N}) of normalized vectors in Y is such that its real linear span is everywhere dense in Y and $1 \leq m(n) \leq k(n) < \infty$ and $m(n) < m(n+1)$ for each $n \in \mathbf{N}$.

Then Y has a Schauder basis.

Proof. Without loss of generality one can select and enumerate

(5) vectors u_1, \dots, u_n so that they are linearly independent in Y for each natural number n . By virtue of Theorem (8.4.8) in [23] their real linear span $\text{span}_{\mathbf{R}}(u_1, \dots, u_n)$ is complemented in Y for each $n \in \mathbf{N}$. Put $L_{n,\infty} := \text{cl}_X \text{span}_{\mathbf{R}}(u_k : k \geq n)$ and $L_{n,m} := \text{cl}_X \text{span}_{\mathbf{R}}(u_k : n \leq k \leq m)$, where $\text{cl}_X A$ denotes the closure of a subset A in X , where $\text{span}_{\mathbf{R}} A$ denotes the real linear span of A . Since Y is a Banach space and $u_k \in Y$ for each k , then $L_{n,\infty} \subset Y$ and $L_{n,m} \subset Y$ for each natural numbers n and m . Then we infer that

$$L_{n,j} \subset \text{span}_{\mathbf{R}}(e_l : m(n) \leq l \leq k_{n,j}), \text{ where } k_{n,j} := \max(k(l) : n \leq l \leq j).$$

Take arbitrary vectors $f \in L_{1,j}$ and $g \in L_{j+1,q}$, where $1 \leq j < q$. Therefore, there are real coefficients f_i and g_i such that

$$f = \sum_{i=1}^{k_{1,j}} f_i e_i \text{ and}$$

$$g = \sum_{i=m(j+1)}^{k_{j+1,q}} g_i e_i. \text{ Hence due to condition (2):}$$

$$\|f - \sum_{i=1}^{m(j)} f_i e_i\|_X \leq s(m(j)) \|f\| \text{ and}$$

$$\|g - \sum_{i=k_{1,j}+1}^{k_{j+1,q}} g_i e_i\|_X \leq s(k_{1,j} + 1) \|g\|_X.$$

On the other hand,

$$f = \sum_{i=1}^{m(j)} f_i e_i + \sum_{i=m(j)+1}^{k_{1,j}} f_i e_i, \text{ consequently,}$$

$$\|f^{[j+1]}\| \leq s(m(j+1)) \|f\|, \text{ where}$$

$$f^{[j+1]} := \sum_{i=m(j+1)}^{k_{1,j}} f_i e_i \text{ and } \sum_{i=a}^b f_i e_i := 0, \text{ when } a > b.$$

When $0 < \delta < 1/4$ and $s(m(j) + 1) < \delta$ we infer using the triangle inequality that $\|f^{[j+1]} - h\|_X \leq \delta \|f^{[j+1]}\|_X / (1 - \delta) \leq \delta s(m(j+1) - 1) \|f\|_X / (1 - \delta)$ for the best approximation h of $f^{[j+1]}$ in $L_{j+1,\infty}$, since $m(j) < m(j+1)$ for each j . Therefore, the inequality $\|f - g\|_X \geq \|f - f^{[j+1]}\|_X - \|f^{[j+1]} - g\|_X$ and $s(n) \downarrow 0$ imply that there exists n_0 such that the inclination of $L_{1,j}$ to $L_{j+1,\infty}$ is not less than $1/2$ for each $j \geq n_0$. Condition (4) implies that L_{1,n_0} is complemented in Y . In virtue of Theorem 1.2.3 [10] a Schauder basis exists in Y .

16. Theorem. *If a set Λ satisfies the Müntz and gap conditions and $1 < p < \infty$, then the Müntz space $M_{\Lambda,p}([0, 1], \mathbf{F})$ has a Schauder basis.*

Proof. In view of Lemma 4 and Theorem 5 there is sufficient to prove an

existence of a Schauder basis in the Müntz space $M_{\Lambda,p}$ for $\Lambda \subset \mathbf{N}$. Mention that if the Müntz space $M_{\Lambda,p}([0,1], \mathbf{R})$ over the real field has the Schauder basis then $M_{\Lambda,p}([0,1], \mathbf{C})$ over the complex field has it as well. Thus it is sufficient to consider the real field $\mathbf{F} = \mathbf{R}$.

Let $U_m(x, Q)$ be kernels of the Fourier summation method in $L_p(0,1)$ as in §2.1 such that

$$(1) \quad \lim_m q_{m,k} = 1 \text{ for each } k \text{ and } \sup_m L_m(Q, L_p) < \infty \text{ and } \sup_{m,k} |q_{m,k}| < \infty.$$

For example, Cesaro's summation method of order 1 can be taken to which Fejér kernels F_n correspond so that the limit

$$\lim_{n \rightarrow \infty} F_n * f = f$$

converges in $L_p(0,1)$ (see Theorem 19.1 and Corollary 19.2 in [33]). That is, there exists a Schauder basis z_n in $L_p(0,1)$ such that

$$z_{2n}(t) = a_{0,2n} + [\sum_{k=1}^{n-1} (a_{k,2n} \cos(2\pi kt) + b_{k,2n} \sin(2\pi kt))] + a_{n,2n} \cos(2\pi nt)$$

and

$$z_{2n+1}(t) = a_{0,2n+1} + \sum_{k=1}^n (a_{k,2n+1} \cos(2\pi kt) + b_{k,2n+1} \sin(2\pi kt))$$

for every $t \in (0,1)$ and $n \in \mathbf{N}$, where $a_{k,j}$ and $b_{k,j}$ are real expansion coefficients.

In virtue of Theorem 6.2.3 and Corollary 6.2.4 [10] each function $g \in M_{\Lambda,p}[0,1]$ has an analytic extension on $\dot{B}_1(0)$ and hence

$$(2) \quad g(z) = \sum_{n=1}^{\infty} c_n z^{\lambda_n} = \sum_{k=1}^{\infty} p_k u_k(z)$$

are the convergent series on the unit open disk $\dot{B}_1(0)$ in \mathbf{C} with center at zero (see §12), where $\Lambda \subset \mathbf{N}$ and $c_n = c_n(g) \in \mathbf{N}$, $p_n = p_n(g) = c_1 + \dots + c_n$, $u_1(z) := z^{\lambda_1}$, $u_{n+1}(z) := z^{\lambda_{n+1}} - z^{\lambda_n}$ for each $n = 1, 2, \dots$. On the other hand, the Müntz spaces $M_{\Lambda,p}[0,1]$ and $M_{\Lambda,p}[\delta^2, 1]$ are isomorphic for each $0 < \delta < 1$ (see Lemma 3 above). Therefore we consider henceforward the Müntz space $M_{\Lambda,p}$ on the segment $[\delta^2, 1]$, where $1 > \delta > 1/2$. Mention that $M_{\Lambda,p}[\delta^2, 1]$ and $M_{\Lambda,p} \circ \sigma[0,1]$ are isomorphic (see §13). Then $Z_{\Lambda,p,2,\delta}$ and $Z_{\Lambda,p,2,\delta} \circ \sigma|_{[0,1]}$ are isomorphic as well. In view of Corollary 9 it is sufficient to prove the existence of a Schauder basis in $Z_{\Lambda,p,2,\delta} \circ \sigma|_{[0,1]}$.

Take the finite dimensional subspace $X_n := \text{span}_{\mathbf{R}}(u_1, \dots, u_n)$ in $M_{\Lambda,p}$, where $n \in \mathbf{N}$. Due to Lemma 4 the Banach space $M_{\Lambda,p} \ominus X_n$ exists and

is isomorphic with $M_{\lambda,p}$. In virtue of Formula I(10.1) [31] $S[(y_{\bar{\beta}_1}^{\psi_1})_{\bar{\beta}_2 - \bar{\beta}_1}^{\psi_2/\psi_1}] = S[y_{\bar{\beta}_2}^{\psi_2}]$, where $y \in L_{\bar{\beta}_2}^{\psi_2}$, when $(\psi_1, \bar{\beta}_1) \leq (\psi_2, \bar{\beta}_2)$.

Consider the trigonometric polynomials $U_m(f, x, Q)$ for $f \in (Z_{\Lambda,p,2,\delta} \ominus (I - Q_2)X_n) \circ \sigma$, where $m = 1, 2, \dots$. Put $Y_{K,n}$ to be the L_p completion of the linear span $\text{span}_{\mathbf{R}}(U_m(f, x, Q) : (m, f) \in K)$, where $K \subset \mathbf{N} \times (Z_{\Lambda,p,2,\delta} \ominus (I - Q_2)X_n) \circ \sigma$, $m \in \mathbf{N}$, $f \in (Z_{\Lambda,p,2,\delta} \ominus (I - Q_2)X_n) \circ \sigma$.

It is known (see Proposition 1.7.1 [31]) that $f \in L_{\beta}^{\psi}(\alpha, \alpha + 1)$ if and only if there exists $g \in L(\alpha, \alpha + 1)$ so that $f = \frac{a_0(f)}{2} + \mathcal{D}_{\psi,\beta} * g$, where the function $\mathcal{D}_{\psi,\beta}$ is prescribed by Formula 6(1), the constant $a_0(f)$ is as above. In view of Lemma 4 it is sufficient to consider the case $a_0(f) = 0$.

There exists a countable subset $\{f_n : n \in \mathbf{N}\}$ in $Z_{\Lambda,p,2,\delta}$ such that $f_n \circ \sigma = \mathcal{D}_{\psi,\beta} * g_n$ with $g_n \in L(0, 1)$ for each $n \in \mathbf{N}$ and so that $\text{span}_{\mathbf{R}}\{f_n : n \in \mathbf{N}\}$ is dense in $Z_{\Lambda,p,\alpha,\delta}$, since $Z_{\Lambda,p,2,\delta}$ is separable. Using Formulas (1, 2), Proposition 12 and Lemma 14 we deduce that a countable set K and a sufficiently large natural number n_0 exist so that the Banach space Y_{K,n_0} is isomorphic with $(Z_{\Lambda,p,2,\delta} \ominus (I - Q_2)X_{n_0})$ and $Y_{K,n_0}|_{(0,1)} \subset W_{\beta}^{\gamma}L_p(0, 1)$, where $0 < \gamma < 1$ and $\beta = 1 - \gamma$. Therefore, by the construction above the Banach space Y_{K,n_0} is the L_p completion of the real linear span of a countable family $(s_l : l \in \mathbf{N})$ of trigonometric polynomials s_l .

Without loss of generality this family can be refined by induction such that s_l is linearly independent of s_1, \dots, s_{l-1} over \mathbf{F} for each $l \in \mathbf{N}$. With the help of transpositions in the sequence $\{s_l : l \in \mathbf{N}\}$, the normalization and the Gaussian exclusion algorithm we construct a sequence $\{r_l : l \in \mathbf{N}\}$ of trigonometric polynomials which are finite real linear combinations of the initial trigonometric polynomials $\{s_l : l \in \mathbf{N}\}$ and satisfying the conditions

$$(3) \|r_l\|_{L_p(0,1)} = 1 \text{ for each } l;$$

(4) the infinite matrix having l -th row of the form $\dots, a_{l,k}, b_{l,k}, a_{l,k+1}, b_{l,k+1}, \dots$ for each $l \in \mathbf{N}$ is upper trapezoidal (step), where

$$r_l(x) = \frac{a_{l,0}}{2} + \sum_{k=m(l)}^{n(l)} [a_{l,k} \cos(2\pi kx) + b_{l,k} \sin(2\pi kx)]$$

with $a_{l,m(l)}^2 + b_{l,m(l)}^2 > 0$ and $a_{l,n(l)}^2 + b_{l,n(l)}^2 > 0$, where $1 \leq m(l) \leq n(l)$, $\deg(r_l) = n(l)$, or $r_l(x) = \frac{a_{l,0}}{2}$ when $\deg(r_l) = 0$; $a_{l,k}, b_{l,k} \in \mathbf{R}$ for each $l \in \mathbf{N}$ and $0 \leq k \in \mathbf{Z}$.

Then as X and Y in Proposition 15 we take $X = L_p[0, 1]$ and $Y = Y_{K, n_0}$. In view of Proposition 15 and Lemma 4 the Schauder basis exists in Y_{K, n_0} and consequently, in $M_{\Lambda, p}$ as well.

References

- [1] I. Al Alam. "A Müntz space having no complement in L_1 ". Proc. Amer. Math. Soc. **136**: **1** (2008), 193-201.
- [2] N.K. Bari. "Trigonometric series" (Oxford: Pergamon Press, 1964).
- [3] P. Borwein, T. Erdélyi. "Polynomials and polynomial inequalities" (New-York: Springer-Verlag, 1995).
- [4] P. Borwein, T. Erdélyi. "Generalizations of Müntz's theorem via a Remez-type inequality for Müntz spaces". J. Amer. Mathem. Soc. **10**: **2** (1997), 327-349.
- [5] N.G. de Bruijn. "Asymptotic methods in Analysis" (Amsterdam: North Holland Publishing Co, 1958).
- [6] J.A. Clarkson, P. Erdős. "Approximation by polynomials". Duke Mathem. J. **10**: **1** (1943), 5-11.
- [7] R.E. Edwards. "Functional Analysis. Theory and applications" (New York: Holt, Rinehart and Winston, 1965).
- [8] G.M. Fichtenholz. "Differential- und Integralrechnung", V. 1-3 (Berlin: VEB Deutscher Verlag für Wissenschaften, 1973).
- [9] M.M. Grinblum. "Some theorems on bases in Banach spaces". Soviet Dokladi 31: 5 (1941), 428-432.
- [10] V.I. Gurariy, W. Lusky. "Geometry of Müntz spaces and related questions". Lecture Notes in Mathematics, **1870** (Berlin: Springer, 2005).
- [11] V.I. Gurariy. "Bases in spaces of continuous functions on compacts and some geometrical questions". Math. USSR. Izvestija **30**: **2** (1966), 289-306.

- [12] H. Jarchow. "Locally convex spaces" (Stuttgart: B.G. Teubner, 1981).
- [13] A.N. Kolmogorov, S.V. Fomin. "Elements of theory of functions and functional analysis" (Moscow: Nauka, 1989).
- [14] J. Lindenstrauss, L. Tzafriri. "Classical Banach spaces"; V. **1, 2** . A series of modern surveys in mathematics **97** (Berlin: Springer-Verlag, 1979).
- [15] S. V. Ludkovsky. " κ -normed topological vector spaces". Siber. Mathem. J. **41: 1** (2000), 141-154.
- [16] S. V. Ludkovsky. "Duality of κ -normed topological vector spaces and their applications". J. Mathem. Sci. (New York: Springer) **157: 2** (2009), 367-385.
- [17] S.V. Ludkowski, W. Lusky. "On the geometry of Müntz spaces". Journal of Function Spaces. online first, DOI 10.1155/2015/787291.
- [18] W. Lusky. "On Banach spaces with the commuting bounded approximation property". Arch. Math. **58: 6** (1992), 568-574.
- [19] W. Lusky. "On Banach spaces with bases". J. Funct. Anal. **138** (1996), 410-425.
- [20] W. Lusky. "Three space properties and basis extensions". Israel. J. Mathem. **107** (1998), 17-27.
- [21] W. Lusky. "Three space problems and bounded approximation property". Stud. Mathem. **159: 3** (2003), 417-434.
- [22] W. Lusky. "On Banach spaces with unconditional bases". Israel. J. Math. **143** (2004), 239-251.
- [23] L. Narici, E. Beckenstein. "Topological vector spaces" (New York: Marcel Dekker, Inc., 1985).
- [24] F.W.J. Olver. "Asymptotics and special functions" (New York: Academic Press, 1974).

- [25] A.P. Prudnikov, Yu.A. Brychkov, O.I. Marichev. "Intergals and series". V.1 (Moscow: Nauka, 1981).
- [26] M. Reed, B. Simon. "Methods of modern mathematical physics". V.2 (New York: Academic Press, 1977).
- [27] B.V. Shabat. "An introduction into complex analysis" (Moscow: Nauka, 1985).
- [28] A.N. Shiriyayev. "Probability" (Moscow: MTzNMO, 2011).
- [29] E.M. Stein. "Singular integrals and diferentiability properties of functions" (Princeton, NJ: Princeton University Press, 1986).
- [30] L. Schwartz. "Étude des sommes d'exponentielles"; 2-ème éd. (Paris: Hermann, 1959).
- [31] A.I. Stepanets. "Classification and approximation of periodic functions", Ser. Mathematics and its applications V. **333** (Dordrecht: Kluwer Acad. Publ., 1995).
- [32] P. Wojtaszczyk. "Banach spaces for analysts". Cambridge studies in advanced mathematics, **25**. (Cambridge: Cambr. Univ. Press, 1991).
- [33] A.C. Zaanen. "Continuity, integration and Fourier theory" (Berlin: Springer, 1989).
- [34] A. Zygmund. "Trigonometric series", V. 1, 2, Third Edition (Cambridge: Cambridge Univ. Press, 2002).